

VERY CLEAN MATRICES OVER LOCAL RINGS

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ABSTRACT. An element $a \in R$ is very clean provided that there exists an idempotent $e \in R$ such that $ae = ea$ and either $a - e$ or $a + e$ is invertible. A ring R is very clean in case every element in R is very clean. We explore the necessary and sufficient conditions under which a triangular 2×2 matrix ring over local rings is very clean. The very clean 2×2 matrices over commutative local rings are completely determined. Applications to matrices over power series are also obtained.

Key Words: very clean ring; very clean matrix; local ring.

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1. INTRODUCTION

A ring R is strongly clean provided that for any $a \in R$ there exist an idempotent $e \in R$ and an element $u \in U(R)$ such that $a = e + u$ and $ae = ea$, where $U(R)$ is the set of all units in R . Recently, strong cleanness has been extensively studied in the literature (cf. [2-3], [5] and [6-8]). We say that an element $a \in R$ is very clean provided that there exists an idempotent $e \in R$ such that $ae = ea$ and either $a - e$ or $a + e$ is invertible. A ring R is very clean in case every element in R is very clean. Clearly, strong cleanness implies the very cleanness. But the converse is not true (see Lemma 2.4). The motivation of this note is to explore very clean matrices over local rings, which also extend weak cleanness from commutative rings to noncommutative rings (cf. [1]). We will construct a large class of very clean rings which are not strongly clean. Let A and B be local rings, let V be an A - B -bimodule, and let $R = \left\{ \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \mid a \in A, b \in B, v \in V \right\}$. We prove that R is very clean if and only if $\frac{1}{2} \in A$ and $\frac{1}{2} \in B$; or R is strongly clean. The characterization of the very cleanness of 2×2 matrices over commutative local rings are completely determined. Let R be a commutative local ring, and let $\varphi \in M_2(R)$. We prove that $\varphi \in M_2(R)$ is very clean if and only if $\frac{1}{2} \in R$; or $\varphi \in M_2(R)$ is strongly clean. Let R be a weakly bleached local ring. We further show that $A(x) \in M_2(R[[x]])$ if and only if $A(0) \in M_2(R)$ is very clean.

Throughout, all rings are associative with an identity. If $\varphi \in M_n(R)$, we use $\chi(\varphi)$ to stand for the characteristic polynomial $\det(tI_n - \varphi)$. $M_n(R)$ and $T_n(R)$ denote the ring of all $n \times n$ matrices and the ring of all $n \times n$ upper triangular matrices over R , respectively.

2. Triangular Matrix Rings

A ring R is local in case it has only one maximal right ideal. As is well known, a ring R is local if and only if $a + b = 1$ in R implies that either a or b is invertible. The purpose of this section is to consider very cleanness for a kind of triangular matrices.

Lemma 2.1. *Let A and B be local rings, let V be an A - B -bimodule, and let*

$$R = \left\{ \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \mid a \in A, b \in B, v \in V \right\}.$$

Then the following are equivalent:

- (1) *R is very clean.*
- (2) *If $a \pm 1 \in J(A)$, $b \in J(B)$ or $a \in J(A)$, $b \pm 1 \in J(B)$, and $v \in V$, there exists $x \in V$ such that $ax - xb = v$.*

Proof. (1) \Rightarrow (2) If $a \pm 1 \in J(A)$, $b \in J(B)$ and $v \in V$, then $r =: \begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} \in R$ is very clean. Thus, we can find an idempotent e_x such that $re_x = e_x r$ and either $r + e_x \in U(R)$ or $r - e_x \in U(R)$, where $e_x = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$. As $re_x = \begin{pmatrix} 0 & ax - v \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & xb \\ 0 & b \end{pmatrix} = e_x r$. Hence $ax - v = xb$, and so $ax - xb = v$.

If $a \in J(A)$, $b \pm 1 \in J(B)$ and $v \in V$, then $r =: \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$ is very clean. Thus, we can find an idempotent f_x such that $rf_x = f_x r$ and either $r + f_x \in U(R)$ or $r - f_x \in U(R)$, where $f_x = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$. As $rf_x = \begin{pmatrix} a & ax \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & v + xb \\ 0 & 0 \end{pmatrix} = f_x r$. Hence, $ax = v + xb$, as required.

(2) \Rightarrow (1) Let $r = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$.

(i) $a \in J(A)$, $b \in J(B)$. Then $r - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U(R)$, and so $r \in R$ is very clean.

(ii) $a \notin J(A)$, $b \notin J(B)$. Then $r \in U(R)$, and so $r \in R$ is very clean.

(iii) $a \notin J(A)$, $b \in J(B)$. If $a \pm 1 \in J(A)$, then there exists $x \in V$ such that $ax - xb = -v$. Hence, $r + \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \in U(R)$. One easily checks that $r \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & ax + v \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & xb \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} r$. Therefore $r \in R$ is very clean.

If $a + 1 \notin J(A)$, then $r + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U(R)$; hence, $r \in R$ is very clean.

If $a - 1 \notin J(A)$, then $r - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U(R)$; hence, $r \in R$ is very clean.

(iv) $a \in J(A), b \notin J(B)$. If $b \pm 1 \in J(B)$, then there exists $x \in V$ such that $ax - xb = v$. Hence, $r + \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \in U(R)$. One easily checks that $r \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & ax \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & v + xb \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} r$. Therefore $r \in R$ is very clean.

If $b + 1 \notin J(B)$, then $r + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U(R)$; hence, $r \in R$ is very clean.

If $b - 1 \notin J(B)$, then $r - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U(R)$; hence, $r \in R$ is very clean. \square

Theorem 2.2. *Let A and B be local rings, let V be an A - B -bimodule, and let*

$$R = \left\{ \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \mid a \in A, b \in B, v \in V \right\}.$$

Then R is very clean if and only if

- (1) $\frac{1}{2} \in A$ and $\frac{1}{2} \in B$; or
- (2) R is strongly clean.

Proof. Suppose that R is very clean. If $\frac{1}{2} \notin A$, then $2 \in J(A)$. If $a \in 1 + J(A), b \in J(B)$, then $a \pm 1 \in J(A)$. By Lemma 2.1, $ax - xb = v$ is solvable. By virtue of [10, Example 2], R is strongly clean. If $\frac{1}{2} \notin B$, then $2 \in J(B)$. If $a \in 1 + J(A), b \in J(B)$, then $a - 1 \in J(A)$. Further, $(b - 1) \pm 1 \in J(B)$. Let $v \in V$. By virtue of Lemma 2.1, $(a - 1)x - x(b - 1) = v$ is solvable. Thus, we can find some $x \in V$ such that $ax - xb = v$. In view of [10, Example 2], R is strongly clean.

We now prove the converse. If $a \pm 1 \in J(A), b \in J(B)$ or $a \in J(A), b \pm 1 \in J(B)$, and $v \in V$, then $2 \in J(A)$ or $2 \in J(B)$, thus $\frac{1}{2} \notin A$ or $\frac{1}{2} \notin B$. This implies that R is strongly clean. Therefore R is very clean, as asserted. \square

Lemma 2.3. *Let R be a commutative ring with exactly two maximal ideals and suppose that $\frac{1}{2} \in R$. Then R is very clean.*

Proof. In view of [2, Proposition 16], for any $a \in R$, there exists an idempotent $e \in R$ such that $ea = ae$ and $a - e \in U(R)$ or $a + e \in U(R)$. Therefore R is very clean. \square

In view of [2, Example 17], $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is a commutative ring with exactly two maximal ideals. We extend this result and derive the following.

Lemma 2.4. *Let $p, q \neq 2$ be prime. If $(p, q) = 1$, then the ring $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ is very clean, but it is not strongly clean.*

Proof. Set $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$. If M is an ideal of R such that $pR \subsetneq M \subseteq R$. Choose $\frac{m}{n} \in M$ while $\frac{m}{n} \notin pR$. Then $p \nmid m$; hence, $(p, m) = 1$. Thus, we can find some $k, l \in \mathbb{Z}$ such that $kp + lm = 1$. Clearly, $\frac{m}{1} = \frac{m}{n} \frac{n}{1} \in M$, hence, $\frac{1}{1} = p \cdot \frac{k}{1} + l \frac{m}{1} \in M$. This implies that $M = R$. Therefore pR is a maximal ideal of R . Likewise, qR is a maximal ideal of R . As $p, q \neq 2$, we see that $p, q \nmid 2$, and so $\frac{1}{2} \in R$. Obviously, $J(R) \subseteq pR \cap qR$. For any $\frac{m}{n} \in pR \cap qR$ and $\frac{a}{b} \in R$, then $\frac{1}{1} - \frac{m}{n} \frac{a}{b} = \frac{nb - ma}{nb}$. Write

$\frac{m}{n} = \frac{ps}{t}$. Then $psn = mt$; hence, $p \mid mt$. As $p \nmid t$, we see that $p \mid m$. Likewise, $q \mid m$. Clearly, $p \nmid nb$, and so $p \nmid (nb - ma)$. Similarly, we see that $q \nmid (nb - ma)$. This yields that $\frac{nb}{nb-ma} \in R$.

Thus, $(\frac{1}{1} - \frac{m}{n} \frac{a}{b}) \cdot \frac{nb}{nb-ma} = \frac{1}{1}$. This means that $\frac{1}{1} - \frac{m}{n} \frac{a}{b} \in U(R)$, and then $\frac{m}{n} \in J(R)$. Therefore $J(R) = pR \cap qR$. Assume that M is a maximal ideal of R and $M \neq pR, qR$. Then $pR + M = R$ and $qR + M = R$. It follows that $R = (pR + M)(qR + M) \subseteq pR \cap qR + M = J(R) + M \subseteq M$, and so $R = M$. This gives a contradiction. Therefore R be a commutative ring with exactly two maximal ideals. According to Lemma 2.3, R is very clean.

As $(p, q) = 1$, we see that $\frac{p(q+1)}{p+q} \in R$. Observing that R is an integral domain, the set of all idempotents in R is $\{\frac{0}{1}, \frac{1}{1}\}$. As $q \nmid q+1$, we see that $\frac{p(q+1)}{p+q} \notin U(R)$. As $p \nmid (p-1)q$, we see that $\frac{p(q+1)}{p+q} - \frac{1}{1} \notin U(R)$. This shows that $\frac{p(q+1)}{p+q} \in R$ is not strongly clean, as required. \square

Theorem 2.5. *The triangular matrix ring $T_2(\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)})$ is very clean, but it is not strongly clean.*

Proof. As $\frac{1}{2} \in \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$, it follows by Theorem 2.2 that $T_2(\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)})$ is very clean, and we therefore complete the proof by Lemma 2.4. \square

3. 2×2 Full Matrices

The aim of this section is to investigate very cleanness of 2×2 full matrices over local rings.

Lemma 3.1. *Let R be a commutative local ring, $\frac{1}{2} \in R$. Then $M_2(R)$ is very clean.*

Proof. Let $\varphi \in M_2(R)$. Write $\chi(\varphi) = t^2 + at + b$. If $\varphi \in GL_2(R)$, or $I_2 - \varphi \in GL_2(R)$; or $I_2 + \varphi \in GL_2(R)$, then φ is very clean. Otherwise, we may assume that $\det(\varphi), \det(I_2 - \varphi), \det(I_2 + \varphi) \in J(R)$. Thus, $b, 1 + a + b, 1 - a + b \in J(R)$. Hence, $2a \in J(R), a \in U(R)$. This implies that $2 \in J(R)$, a contradiction. Therefore we conclude that $M_2(R)$ is very clean. \square

Let $R = \mathbb{Z}[x]_{(x)} = \{\frac{f(x)}{g(x)} \mid g(0) \neq 0\}$ be the localization of $\mathbb{Z}[x]$ at (x) . Then R is a commutative local ring with $\frac{1}{2} \in R$. It follows by Lemma 3.1 that $M_2(R)$ is very clean.

Lemma 3.2. *Let R be a ring, $2 \in J(R)$, and let $a \in R$. Then a is very clean if and only if a is strongly clean.*

Proof. If $a \in R$ is strongly clean, then it is very clean. Conversely, assume that $a \in R$ is very clean. Then there exist an idempotent $e \in R$ and a unit $u \in R$ such that $ae = ea$ and either $a = e + u$ or $a = -e + u$. If $a = -e + u$, then $a = e + (u - 2e)$.

As $2 \in J(R)$, we see that $u - 2e \in U(R)$; hence, $a \in R$ is strongly clean. Therefore we complete the proof. \square

Theorem 3.3. *Let R be a commutative local ring, and let $\varphi \in M_2(R)$. Then $\varphi \in M_2(R)$ is very clean if and only if*

- (1) $\frac{1}{2} \in R$; or
- (2) $\varphi \in M_2(R)$ is strongly clean.

Proof. Assume that $\varphi \in M_2(R)$ is very clean. If $\frac{1}{2} \notin R$, then $2 \in J(R)$. Thus, $2I_2 \in J(M_2(R))$. In view of Lemma 3.2, $\varphi \in M_2(R)$ is strongly clean.

Conversely, if $\frac{1}{2} \in R$, it follows from Lemma 3.1 that $\varphi \in M_2(R)$ is very clean. If $\varphi \in M_2(R)$ is strongly clean, then it is very clean. Therefore φ is very clean in any case. \square

Example 3.4. *Let $p \in \mathbb{Z}$ be a prime, and $p \neq 2$. Then $\begin{pmatrix} 1 & p \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(p)})$ is very clean, while it is not strongly clean.*

Proof. As $p \neq 2$, $(2, p) = 1$, we can find some $k, l \in \mathbb{Z}$ such that $2k + pl = 1$; hence, $2 \in U(\mathbb{Z}_{(p)})$. In view of Theorem 3.3, $M_2(\mathbb{Z}_{(p)})$ is a very clean ring, and so $\begin{pmatrix} 1 & p \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(p)})$ is very clean. But it is not strongly clean from [5, Corollary 16.4.33]. \square

For $r \in R$, define $\mathbb{S}_r = \{f \in R[t] \mid f \text{ monic, and } f(r) \in U(R)\}$.

Lemma 3.5. *Let R be a commutative local ring, $n \geq 2$, and let $h \in R[t]$ be a monic polynomial of degree n . Then the following are equivalent:*

- (1) Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is very clean.
- (2) There exists a factorization $h = h_0 h_1$ such that $(h_0, h_1) = R[t]$ and $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1 \cup \mathbb{S}_{-1}$.

Proof. (1) \Rightarrow (2) Since φ is very clean, we see that φ or $-\varphi$ is strongly clean. If φ is strongly clean, it follows by [4, Theorem 12] that there exists a factorization $h = h_0 h_1$ such that $(h_0, h_1) = R[t]$ and $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1$.

If $-\varphi$ is strongly clean, it follows by [4, Theorem 12] that $g(\mu) := \det(\mu I_n - (-\varphi)) = g_0 g_1$ where $(g_0, g_1) = R[\mu]$ and $g_0 \in \mathbb{S}_0$ and $g_1 \in \mathbb{S}_1$. This implies that

$$h(t) = \det(tI_n - \varphi) = (-1)^n g(-t) = (-1)^n g_0(-t) g_1(-t).$$

Set $h_0 = (-1)^{\deg g_0} g_0(-t)$ and $h_1 = (-1)^{\deg g_1} g_1(-t)$. Then $h = h_0 h_1$ with $(h_0, h_1) = R[t]$. Further, $h_0(0) = (-1)^{\deg g_0} g_0(0) \in U(R)$; hence, $h_0 \in \mathbb{S}_0$. In addition, $h_1(-1) = (-1)^{\deg g_1} g_1(1) \in U(R)$. This implies that $h_1 \in \mathbb{S}_{-1}$. Therefore $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1 \cup \mathbb{S}_{-1}$.

(2) \Rightarrow (1) By hypothesis, there exists a factorization $h = h_0 h_1$ such that $(h_0, h_1) = R[t]$ and $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1 \cup \mathbb{S}_{-1}$. If $h_1 \in \mathbb{S}_1$, it follows by [4, Theorem 12] that $\varphi \in M_n(R)$ is strongly clean, and so it is very clean. If $h_1 \in \mathbb{S}_{-1}$, then $h_1(-1) \in U(R)$. Set $g(\mu) := (-1)^n h(-\mu)$. Then $g(\mu) = g_0 g_1$ where

$g_0(\mu) = (-1)^{\deg h_0} h_0(-\mu)$ and $g_1(\mu) = (-1)^{\deg h_1} h_1(-\mu)$. As $g_0(0) \in U(R)$, we see that $g_0 \in \mathbb{S}_0$. Further, $g_1(1) = (-1)^{\deg h_1} h_1(-1) \in U(R)$, and then $g_1 \in \mathbb{S}_1$. Clearly, $g(\mu) = \det(\mu I_n - (-\varphi))$. In view of [4, Theorem 12], $-\varphi \in M_n(R)$ is strongly clean. Therefore $\varphi \in M_n(R)$ is very clean, as asserted. \square

In what follows, we consider more explicit criteria for very clean 2×2 matrices over commutative rings.

Theorem 3.6. *Let R be a commutative local ring, and let $h \in R[t]$ be a monic polynomial of degree 2. Then the following are equivalent:*

- (1) *Every $\varphi \in M_2(R)$ with $\chi(\varphi) = h$ is very clean.*
- (2) *There exists a factorization $h = h_0 h_1$ such that $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1 \cup \mathbb{S}_{-1}$.*

Proof. (1) \Rightarrow (2) is trivial from Lemma 3.5.

(2) \Rightarrow (1) By hypothesis, there exists a factorization $\chi(\varphi) = h_0 h_1$ such that $h_0 \in \mathbb{S}_0$ and $h_1 \in \mathbb{S}_1 \cup \mathbb{S}_{-1}$.

Case I. $\deg(h_0) = 2$ and $\deg(h_1) = 0$. Then $h_0 = t^2 - \text{tr}(\varphi)t + \det(\varphi)$ and $h_1 = 1$. Hence, $(h_0, h_1) = R[t]$.

Case II. $\deg(h_0) = 1$ and $\deg(h_1) = 1$. Then $h_0 = t - \alpha$ and $h_1 = t - \beta$. Since $h_0 \in \mathbb{S}_0$, $\alpha \in U(R)$. As $h_1 \in \mathbb{S}_1 \cup \mathbb{S}_{-1}$, we see that $\beta \in 1 + U(R)$ or $\beta \in -1 + U(R)$.

If $\beta - \alpha \in U(R)$, then $h_0(\beta - \alpha)^{-1} - h_1(\beta - \alpha)^{-1} = 1$; hence, $(h_0, h_1) = R[t]$.

If $\beta - \alpha \in J(R)$, then $\beta \in U(R)$. Let $h'_0 = t^2 - (\alpha + \beta)t + \alpha\beta$ and $h'_1 = 1$. Then $h'_0 \in \mathbb{S}_0$ and $h'_1 \in \mathbb{S}_1$. In addition, $(h'_0, h'_1) = R[t]$.

Case III. $\deg(h_0) = 0$ and $\deg(h_1) = 2$. Then $h_0 = 1$ and $h_1 = \det(tI_2 - \varphi)$. Hence, $(h_0, h_1) = R[t]$.

In any case, there exists a factorization $h = h_0 h_1$ such that $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1 \cup \mathbb{S}_{-1}$ and $(h_0, h_1) = R[t]$. Therefore we complete the proof by Lemma 3.5. \square

Corollary 3.7. *Let R be a commutative local ring, and let $\varphi \in M_2(R)$. Then φ is very clean if and only if $\varphi \in M_2(R)$ is strongly clean; or $I_2 + \varphi \in GL_2(R)$.*

Proof. Assume that $\varphi \in M_2(R)$ is strongly clean; or $I_2 + \varphi \in GL_2(R)$. Then φ is very clean. Conversely, assume that φ is very clean and $I_2 + \varphi \notin GL_2(R)$. We may assume that $\varphi, I_2 - \varphi \notin GL_2(R)$, and so $\det(\varphi), \det(I_2 - \varphi), \det(I_2 + \varphi) \in J(R)$. It follows from $\det(\varphi) = -\alpha(\beta + a) \in J(R)$ that $\beta + a \in J(R)$. Hence, $1 + \beta + a \in U(R)$. Set $h_0 = t - \alpha$ and $h_1 = t + \beta + a$. Then $h_0 \in \mathbb{S}_0, h_1 \in \mathbb{S}_1$. As in the proof of Theorem 3.6, we may assume that $(h_0, h_1) = 1$. In light of [4, Theorem 12], φ is clean, as required. \square

4. MATRICES OVER POWER SERIES

Let $a \in R$. Then $l_a : R \rightarrow R$ and $r_a : R \rightarrow R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for

any $r \in R$. Following Diesl, a local ring R is weakly bleached provided that for any $a \in 1 + J(R)$, $b \in J(R)$, $\ell_a - r_b, \ell_b - r_a : R \rightarrow R$ are surjective. The class of weakly bleached local rings contains many familiar examples, e.g., commutative local rings, local rings with nil Jacobson radicals, local rings for which some power of each element of their Jacobson radicals is central (cf. [3, Example 13]). The goal of this section is to investigate very clean matrices with power series entries over local rings.

Lemma 4.1. *Let R be a ring. Then $f(x) \in R[[x]]$ is very clean if there exists an idempotent $e \in R$ such that*

- (1) $f(0)e = ef(0)$;
- (2) $f(0) - e \in U(R)$ or $f(0) + e \in U(R)$;
- (3) for any $b \in R$, there exists an $x \in R$ such that $[f(0), x] = [e, b]$.

Proof. By hypothesis, there exists an idempotent $e \in R$ such that $f(0)e = ef(0)$ and either $f(0) - e \in U(R)$ or $f(0) + e \in U(R)$. Assume that $f(0) - e \in U(R)$. For any $b \in R$, we can find an $x \in R$ such that $[f(0), x] = [e, b]$. Applying [8, Theorem 3.3.2] to $f(x)$, we see that $f(x) \in R[[x]]$ is strongly clean. If $f(0) + e \in U(R)$, then $-f(0) - e \in U(R)$. For any $b \in R$, we can find an $x \in R$ such that $[f(0), x] = [e, b]$. Choose $y = -x$. Then $[-f(0), y] = [e, b]$. Applying [8, Theorem 3.3.2] to $-f(x)$, we see that $-f(x) \in R[[x]]$ is strongly clean. Therefore $f(x)$ is very clean, as required. \square

Theorem 4.2. *Let R be a weakly bleached local ring. Then the following are equivalent:*

- (1) $A(x) \in M_2(R[[x]])$ is very clean.
- (2) $A(0) \in M_2(R)$ is very clean.

Proof. (1) \Rightarrow (2) Since $A(x)$ is very clean in $M_2(R[[x]])$, there exist an $E(x) = E^2(x) \in M_2(R[[x]])$ and a $U(x) \in GL_2(R[[x]])$ such that $E(x)U(x) = U(x)E(x)$, and that either $A(x) = E(x) + U(x)$ or $A(x) = -E(x) + U(x)$. This implies that $E(0)U(0) = U(0)E(0)$, and that either $A(0) = E(0) + U(0)$ or $A(0) = -E(0) + U(0)$, where $E(0) = E^2(0) \in M_2(R)$ and $U(0) \in GL_2(R)$. As a result, $A(0)$ is very clean in $M_2(R)$.

(2) \Rightarrow (1) Clearly, $M_2(R[[x]]) \cong M_2(R)[[x]]$. We may assume that $A(x) \in M_2(R)[[x]]$. Set $S = M_2(R)$. Then $A(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$. Then there exists an idempotent $e = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \in S$ such that $A(0)e = eA(0)$ and either $A(0) - e \in U(S)$ or $A(0) + e \in U(S)$. In view of [5, Lemma 16.4.10], there exists some $u \in U(S)$ such that $ueu^{-1} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$. Clearly, $e_1 = e_1^2, e_2 = e_2^2$. As R is local, e_1 and e_2 are trivial idempotents. If $e_1 = e_2 = 0$ or $e_1 = e_2 = 1$, then $e = 0$ or $e = I_2$, and so for any $s \in S$, there exists an $x = 0$ such that $[A(0), x] = [e, s]$. Thus, we may assume that $e_1 = 1$ and $e_2 = 0$. It follows from $A(0)e = eA(0)$ that $(uA(0)u^{-1})(ueu^{-1}) = (ueu^{-1})(uA(0)u^{-1})$; hence, $uA(0)u^{-1} =$

$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$, where $a_{22} \in U(R)$ and either $a_{11} \in 1 + U(R)$ or $a_{11} \in -1 + U(R)$.

Set $\alpha := \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$. Assume that $a_{11} \in 1 + U(R)$ and $a_{22} \in U(R)$. If $a_{22} \in 1 + U(R)$, then we choose $f = I_2$, then $\alpha - f \in U(S)$, $\alpha f = f\alpha$ and that for any $\beta \in S$, $[\alpha, 0] = [f, \beta]$. If $a_{11} \in U(R)$, then we choose $f = 0$, then $\alpha - f \in U(S)$, $\alpha f = f\alpha$ and that for any $\beta \in S$, $[\alpha, 0] = [f, \beta]$. Thus, we assume that $a_{11} \in J(R)$, $a_{22} \in 1 + J(R)$. Choose $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Then $\alpha - f \in U(S)$, $\alpha f = f\alpha$. For any $\beta = (\beta_{ij}) \in S$, as R is weakly bleached, there exist some $x_1, x_2 \in S$ such that $a_{11}x_1 - x_1a_{22} = \beta_{12}$ and $a_{22}x_2 - x_2a_{11} = -\beta_{21}$. Choose $x = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} \in R$. It is easy to verify that

$$[\alpha, x] = \begin{pmatrix} 0 & a_{11}x_1 - x_1a_{22} \\ a_{22}x_2 - x_2a_{11} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta_{12} \\ -\beta_{21} & 0 \end{pmatrix} = [f, \beta].$$

Assume that $a_{11} \in -1 + U(R)$ and $a_{22} \in U(R)$. If $a_{22} \in -1 + U(R)$, then we choose $f = I_2$, then $\alpha + f \in U(S)$, $\alpha f = f\alpha$ and that for any $\beta \in S$, $[\alpha, 0] = [f, \beta]$. If $a_{11} \in U(R)$, then we choose $f = 0$, then $\alpha - f \in U(S)$, $\alpha f = f\alpha$ and that for any $\beta \in S$, $[\alpha, 0] = [f, \beta]$. Thus, we assume that $a_{11} \in J(R)$, $a_{22} \in -1 + J(R)$. Thus, $-a_{11} \in J(R)$, $-a_{22} \in 1 + J(R)$. Choose $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$. Then $\alpha + f \in U(S)$, $\alpha f = f\alpha$. For any $\beta = (\beta_{ij}) \in S$, as R is weakly bleached, there exist some $x_1, x_2 \in R$ such that $(-a_{11})x_1 - x_1(-a_{22}) = -\beta_{12}$ and $(-a_{22})x_2 - x_2(-a_{11}) = \beta_{21}$. Choose $x = \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} \in S$. Then we check that $[\alpha, x] = \begin{pmatrix} 0 & a_{11}x_1 - x_1a_{22} \\ a_{22}x_2 - x_2a_{11} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta_{12} \\ -\beta_{21} & 0 \end{pmatrix} = [f, \beta]$. The case $e_1 = 0$, $e_2 = 1$ is similar.

For any $s \in S$, it follows from the preceding discussion that there exists an $x \in S$ such that $[uA(0)u^{-1}, x] = [ueu^{-1}, usu^{-1}]$. Therefore $[A(0), u^{-1}xu] = [e, s]$. According to Lemma 4.1, $A(x)$ is very clean. \square

Let $p (\neq 2)$ be a prime number. In light of Theorem 4.2, the ring $M_2(\mathbb{Z}_{(p)}[[x]])$ is very clean. But $M_2(\mathbb{Z}_{(p)}[[x]])$ is not strongly clean.

Corollary 4.3. *Let R be a commutative local ring, and let $A(x) \in M_2(R[[x]])$. Then the following are equivalent:*

- (1) $A(x) \in M_2(R[[x]])$ is very clean.
- (2) $A(0) \in M_2(R)$ is very clean.

Proof. Since every commutative local ring is weakly bleached, we complete the proof by Theorem 4.2. \square

Corollary 4.4. *Let R be a commutative local ring, and let $A(x) \in M_2(R[[x]]/(x^m))$ ($m \geq 1$). Then the following are equivalent:*

- (1) $A(x) \in M_2(R[[x]]/(x^m))$ is very clean.

(2) $A(0) \in M_2(R)$ is very clean.

Proof. It is obvious from Corollary 4.3. \square

Example 4.5. Let $R = \mathbb{Z}_4[x]/(x^2)$, and let

$$A(x) = \begin{pmatrix} \bar{3} & \bar{2} + \bar{2}x \\ \bar{2} + x & \bar{3}x \end{pmatrix} \in M_2(R).$$

Obviously, \mathbb{Z}_4 is a commutative local ring, and that $R = \mathbb{Z}_4[[x]]/(x^2)$. Since we have the very clean decomposition $A(0) = \begin{pmatrix} \bar{3} & \bar{0} \\ \bar{0} & \bar{3} \end{pmatrix} + \begin{pmatrix} \bar{0} & \bar{2} \\ \bar{2} & \bar{1} \end{pmatrix}$ in $M_2(\mathbb{Z}_4)$, it follows by Corollary 4.4 that $A(x) \in M_2(R)$ is very clean. \square

Theorem 4.6. Let R be a weakly bleached local ring. Then the following are equivalent:

- (1) $A(x) \in T_2(R[[x]])$ is very clean.
- (2) $A(0) \in T_2(R)$ is very clean.

Proof. (1) \Rightarrow (2) is obvious.

$$(2) \Rightarrow (1) \text{ Let } A(0) = \begin{pmatrix} a_1 & c \\ 0 & a_2 \end{pmatrix} \in R \text{ and } S = T_2(R).$$

(a) $a_1, a_2 \in 1 + U(R)$. Then we choose $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $A(0) - e \in U(S)$ and $A(0)e = eA(0)$. For any $b \in S$, we choose $x = 0$. Then $[A(0), x] = [e, b]$.

(b) $a_1, a_2 \in -1 + U(R)$. Then we choose $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $A(0) + e \in U(S)$ and $A(0)e = eA(0)$. For any $b \in S$, we choose $x = 0$. Then $[A(0), x] = [e, b]$.

(c) $a_1, a_2 \in U(R)$. Then $A(0) \in U(S)$, and so we choose $e = 0$. Then $A(0) - e \in U(S)$ and $A(0)e = eA(0)$. For any $b \in S$, we choose $x = 0$. Then $[A(0), x] = [0, b]$.

Thus, either $a_1 \in 1 + J(R), a_2 \in J(R)$ or $a_1 \in J(R), a_2 \in 1 + J(R)$, and that either $a_1 \in -1 + J(R), a_2 \in J(R)$ or $a_1 \in J(R), a_2 \in -1 + J(R)$. Therefore we may assume that either $a_1 \in \pm 1 + J(R), a_2 \in J(R)$ or $a_1 \in J(R), a_2 \in \pm 1 + J(R)$. For such $a \in S$, by hypothesis, there exist an idempotent $e \in S$ and a unit $u \in S$ such that $A(0)e = eA(0)$ and either $a - e = u$ or $a + e = u$.

Case I. $a_1 \in \pm 1 + J(R), a_2 \in J(R)$. Then $e = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix}$. For any $b = \begin{pmatrix} b_1 & z \\ 0 & b_2 \end{pmatrix} \in S$, as R is weakly bleached, we can find a $w \in R$ such that $a_1w - wa_2 = yb_2 - b_1y - z$. Choose $x = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \in S$. Then $[A(0), x] = \begin{pmatrix} 0 & a_1w - wa_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & yb_2 - b_1y - z \\ 0 & b_2 \end{pmatrix} = [e, b]$.

Case II. $a_1 \in J(R), a_2 \in \pm 1 + J(R)$. Then $e = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$. For any $b = \begin{pmatrix} b_1 & z \\ 0 & b_2 \end{pmatrix} \in S$, as R is weakly bleached, we can find a $w \in R$ such that $a_1w -$

$wa_2 = z + yb_2 - b_1y$. Choose $x = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \in S$. Then

$$[A(0), x] = \begin{pmatrix} 0 & a_1w - wa_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z + yb_2 - b_1y \\ 0 & 0 \end{pmatrix} = [e, b].$$

Thus, $A(0)$ satisfies the conditions in Lemma 4.1. Since $T_2(R[[x]]) \cong T_2(R)[[x]]$, we conclude that $A(x)$ is very clean. \square

Corollary 4.7. *Let R be a local ring. Then the following are equivalent:*

- (1) $T_2(R)$ is very clean.
- (2) $T_2(R[[x]])$ is very clean.

Proof. (1) \Rightarrow (2) Since $T_2(R)$ is very clean, it follows from Theorem 2.2 that $\frac{1}{2} \in R$ or $T_2(R)$ is strongly clean. If $\frac{1}{2} \in R$, as in the proof of Theorem 2.2, $T_2(R)$ is very clean. If $T_2(R)$ is strongly clean, it follows from [10, Example 2] that R is weakly bleached. Therefore $T_2(R[[x]])$ is very clean by Theorem 4.6.

(2) \Rightarrow (1) is just as easy. \square

Let R be a commutative local ring. If $\frac{1}{2} \in R$, then $T_2(R[[x]])$ is very clean. As in the proof of Theorem 2.2, $T_2(R)$ is very clean. Therefore we are done by Corollary 4.7. For instance, for any prime p ($\neq 2$). Then $T_2(\mathbb{Z}_{(p)}[[x]])$ is very clean.

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